Sean las sucesiones $a_n = \frac{1-(-1)^n}{2}$ y $b_n = \frac{n-a_n}{2}$. Entonces $b_n = \frac{2n-1+(-1)^n}{4}$.

Por ejemplo, para $x \in \mathbb{R}$, $\frac{d^n}{dx^n}(\sin x) = (-1)^{b_n}(a_n \cos x + (1-a_n)\sin x)$

Por otro lado, haciendo uso de las fórmulas de Eüler y de De Moivre, y del teorema del binomio, se obtiene:

$$e^{inx} = \cos nx + i\sin nx = (\cos x + i \sin x)^n = \sum_{k=0}^{n} \binom{n}{k} (\cos x)^{n-k} (i \sin x)^k \qquad (0)$$

Fórmulas de ángulo múltiple para el coseno

De (0), se obtiene:

$$\cos nx = \sum_{k=0}^{b_n} \binom{n}{2k} (-1)^k (\cos x)^{n-2k} (\sin x)^{2k}$$

$$= \sum_{k=0}^{b_n} \binom{n}{2k} \left(\sum_{p=0}^{k} \binom{k}{p} (-1)^p (\cos x)^{n-2p} \right)$$

$$= \sum_{k=0}^{b_n} \left(\sum_{p=k}^{b_n} \binom{n}{2p} \binom{p}{k} \right) (-1)^k (\cos x)^{n-2k} \qquad (1)$$

Observación:

$$\sum_{k=0}^{n} u_k \left(\sum_{p=0}^{k} v_p w_{p,k} \right) = \sum_{k=0}^{n} \left(\sum_{p=k}^{n} u_p w_{k,p} \right) v_k$$

Luego,

$$\sum_{n=0}^{m} \cos nx = \sum_{n=0}^{m} \left(\sum_{k=0}^{b_{m-a_n}-b_n} \left(\sum_{p=k}^{b_{2k+n}} \binom{2k+n}{2p} \binom{p}{k} \right) (-1)^k \right) (\cos x)^n = \frac{\sin(m+1)\frac{x}{2}}{\sin\frac{x}{2}} \cos m\frac{x}{2} \qquad (2)$$

Observación:

$$\sum_{n=0}^{m} \left(\sum_{k=0}^{b_n} u_k v_{n-2k} w_{n,k} \right) = \left[\sum_{n=0}^{b_m} \left(\sum_{k=0}^{b_m-n} u_k w_{2k+2n,k} \right) v_{2n} \right] + \left[\sum_{n=0}^{b_{m-1}} \left(\sum_{k=0}^{b_{m-1}-n} u_k w_{2k+(2n+1),k} \right) v_{2n+1} \right] = \sum_{n=0}^{m} \left(\sum_{k=0}^{b_{m-a_n}-b_n} u_k w_{2k+n,k} \right) v_n$$

Consideremos de (1):

$$\cos nx = (\cos x)^n \sum_{p=0}^{b_n} \binom{n}{2p} + \sum_{k=1}^{b_n} \left(\sum_{p=k}^{b_n} \binom{n}{2p} \binom{p}{k} \right) (-1)^k (\cos x)^{n-2k}$$

Por consiguiente,

$$(\cos x)^n = \frac{1}{2^{n-1}}\left(\cos nx - \sum_{k=1}^{b_n}\left(\sum_{p=k}^{b_n}\binom{n}{2p}\binom{p}{k}\right)(-1)^k(\cos x)^{n-2k}\right)$$

Observación:

$$\sum_{p=0}^{b_n}\binom{n}{2p} = 2^{n-1}$$

Ahora, usando (2) se obtiene:

$$\sum_{m=0}^{q}(\cos mx)^n = \frac{1}{2^{n-1}}\left(\frac{\sin(q+1)\frac{nx}{2}}{\sin\frac{nx}{2}}\cos q\frac{nx}{2} - \sum_{m=0}^{q}\sum_{k=1}^{b_n}(-1)^k\left(\sum_{p=k}^{b_n}\binom{n}{2p}\binom{p}{k}\right)(\cos mx)^{n-2k}\right)$$

$$= \frac{1}{2^{n-1}}\left(\frac{\sin(q+1)\frac{nx}{2}}{\sin\frac{nx}{2}}\cos q\frac{nx}{2} - \sum_{k=1}^{b_n}(-1)^k\left(\sum_{p=k}^{b_n}\binom{n}{2p}\binom{p}{k}\right)\left(\sum_{m=0}^{q}(\cos mx)^{n-2k}\right)\right) \quad (*)$$

Observación:

$$\sum_{m=0}^{q}\left(\sum_{k=1}^{n}u_k v_{m,k}\right) = \sum_{k=1}^{n}u_k\left(\sum_{m=0}^{q}v_{m,k}\right)$$

Fórmulas de ángulo múltiple para el seno

De (0), se obtiene:

$$\sin nx = \sum_{k=0}^{b_{n-1}}(-1)^k\binom{n}{2k+1}(\cos x)^{n-(2k+1)}(\sin x)^{2k+1}$$

$$= (\cos x)^{a_{n-1}}\sum_{k=0}^{b_{n-1}}\binom{n}{2k+1}\left(\sum_{p=k}^{b_{n-1}}(-1)^p\binom{b_{n-1}-k}{b_{n-1}-p}(\sin x)^{2p+1}\right)$$

$$= (\cos x)^{a_{n-1}}\sum_{k=0}^{b_{n-1}}\left(\sum_{p=0}^{k}\binom{n}{2p+1}\binom{b_{n-1}-p}{b_{n-1}-k}\right)(-1)^k(\sin x)^{2k+1} \qquad (3)$$

Observación:

$$\sum_{k=0}^{n}u_k\left(\sum_{p=k}^{n}v_p w_{p,k}\right) = \sum_{k=0}^{n}\left(\sum_{p=0}^{k}u_p w_{k,p}\right)v_k$$

Luego,

$$\sum_{n=0}^{m}\sin nx = \sum_{n=0}^{b_{m-1}}(-1)^n(\sin x)^{2n+1}\left(\sum_{k=2n+1}^{m}(\cos x)^{a_{n-1}}\left(\sum_{p=0}^{n}\binom{k}{2p+1}\binom{b_{k-1}-p}{b_{k-1}-n}\right)\right) = \frac{\sin(m+1)\frac{x}{2}}{\sin\frac{x}{2}}\sin m\frac{x}{2}$$

Observación:

$$\sum_{n=0}^{m} u_n \left(\sum_{k=0}^{b_{n-1}} v_k w_{n,k} \right) = \sum_{n=0}^{b_{m-1}} \left(\sum_{k=2n+1}^{m} u_k w_{k,n} \right) v_n \wedge u_0 = 0$$

Consideremos de (3):

$$\sin(2n+1)x = \sum_{k=0}^{n} \left(\sum_{p=0}^{k} \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k (\sin x)^{2k+1} \qquad (4)$$

Si se hace $x = \frac{\pi}{2n+1}$, se obtiene:

$$\sum_{k=0}^{n} \left(\sum_{p=0}^{k} \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k \left(\sin \frac{\pi}{2n+1} \right)^{2k} = 0$$

Si $n = 2$, se obtiene la expresión $16 \left(\sin \frac{\pi}{5} \right)^4 - 20 \left(\sin \frac{\pi}{5} \right)^2 + 5 = 0$. Puesto que $\sin \frac{\pi}{6} < \sin \frac{\pi}{5} < \sin \frac{\pi}{4}$, entonces $\sin \frac{\pi}{5} = \frac{1}{2} \sqrt{\frac{5-\sqrt{5}}{2}}$.

Por otro lado, de (4):

$$\sum_{n=0}^{b_m} \sin(2n+1)x = \sum_{n=0}^{b_m} \left(\sum_{k=n}^{b_m} \left(\sum_{p=0}^{n} \binom{2k+1}{2p+1} \binom{k-p}{k-n} \right) \right) (-1)^n (\sin x)^{2n+1} = \frac{(\sin b_m x)^2}{\sin x} \qquad (5)$$

Observación:

$$\sum_{n=0}^{m} \left(\sum_{k=0}^{n} u_k v_{n,k} \right) = \sum_{n=0}^{m} \left(\sum_{k=n}^{m} v_{k,n} \right) u_n$$

Consideremos de (4):

$$\sin(2n+1)x = (-1)^n (\sin x)^{2n+1} \sum_{p=0}^{n} \binom{2n+1}{2p+1} + \sum_{k=0}^{n-1} \left(\sum_{p=0}^{k} \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k (\sin x)^{2k+1}$$

Por consiguiente,

$$(\sin x)^{2n+1} = \frac{(-1)^n}{2^{2n}} \left(\sin(2n+1)x - \sum_{k=0}^{n-1} \left(\sum_{p=0}^{k} \binom{2n+1}{2p+1} \binom{n-p}{n-k} \right) (-1)^k (\sin x)^{2k+1} \right)$$

Observación:

$$\sum_{p=0}^{n} \binom{2n+1}{2p+1} = 2^{2n}$$

Ahora, usando (5) se obtiene:

$$\sum_{m=0}^{q}(\sin mx)^{2n+1}=\frac{(-1)^n}{2^{2n}}\left(\frac{\sin(q+1)\frac{(2n+1)x}{2}}{\sin\frac{(2n+1)x}{2}}\sin q\,\frac{(2n+1)x}{2}-\sum_{m=0}^{q}\left(\sum_{k=0}^{n-1}\left(\sum_{p=0}^{k}\binom{2n+1}{2p+1}\binom{n-p}{n-k}\right)(-1)^k(\sin mx)^{2k+1}\right)\right)$$

$$=\frac{(-1)^n}{2^{2n}}\left(\frac{\sin(q+1)\frac{(2n+1)x}{2}}{\sin\frac{(2n+1)x}{2}}\sin q\,\frac{(2n+1)x}{2}-\sum_{k=0}^{n-1}(-1)^k\left(\sum_{p=0}^{k}\binom{2n+1}{2p+1}\binom{n-p}{n-k}\right)\left(\sum_{m=0}^{q}(\sin mx)^{2k+1}\right)\right)\quad(**)$$

Observación:

$$\sum_{m=0}^{q}\left(\sum_{k=0}^{n}u_k v_{m,k}\right)=\sum_{k=0}^{n}u_k\left(\sum_{m=0}^{q}v_{m,k}\right)\wedge v_{0,k}=0$$

Finalmente, hallamos que las expresiones $(*)$ y $(**)$ son similares a la recurrencia entre coeficientes de la fórmula polinómica general de la suma de las n-ésimas y $2n+1$-ésimas potencias, respectivamente, de los primeros q naturales.